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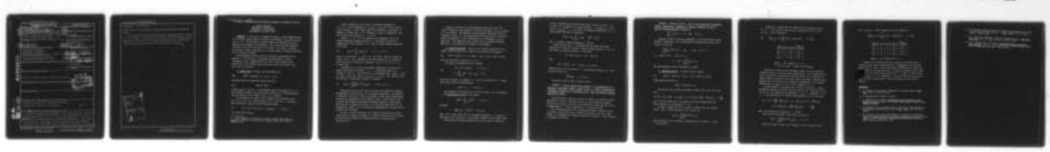

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(continued) 20. ABSTRACT

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The Lax-Wendroff scheme and other dissipative approximations are applied to a test problem. As expected from Gustafsson's rate-of-convergence theory, computations verify that if the boundary extrapolation and the difference scheme have equal order of accuracy, then this order is preserved.

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ON BOUNDARY EXTRAPOLATION AND DISSIPATIVE SCHEMES FOR HYPERBOLIC PROBLEMS

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ABSTRACT. In this note we consider dissipative, stable approximations to well-posed linear hyperbolic initial value problems in the quarter plane $x \geq 0$, $t \geq 0$. We show that if boundary values are determined by extrapolation, then stability is maintained. This result was first suggested by Kreiss, and proved explicitly by the author. The proof is reviewed here using a stability criterion for a certain family of boundary conditions due to Goldberg and Tadmor.

The Lax-Wendroff scheme and other dissipative approximations are applied to a test problem. As expected from Gustafsson's rate-of-convergence theory, computations verify that if the boundary extrapolation and the difference scheme have equal order of accuracy, then this order is preserved.

1. INTRODUCTION.

Consider the conservation law

$$\partial u/\partial t + \partial g(u)/\partial x = 0, \quad x \geq 0, \quad t \geq 0, \tag{1a}$$

and assume that the associated initial value pro

$$u(x,0) = f(x) \tag{1b}$$

is well-posed in $L^2(0,\infty)$, so no boundary values are required at $x = 0$. This assumption implies that characteristic lines do not carry information from the exterior of the domain $x \geq 0$, $t \geq 0$ inward.

To approximate the initial value problem (1), we introduce a mesh size $\Delta x > 0$, $\Delta t > 0$; a grid function $v_\nu(t) = v(\nu\Delta x, t)$, $\nu = 0, \pm 1, \pm 2, \ldots$; and a consistent, explicit finite difference scheme

$$v_\nu(t + \Delta t) = S\{v_{\nu-r}(t), \ldots, v_{\nu+p}(t)\}; \quad \nu = 1,2,3,\ldots, \tag{2}$$

r,p being fixed integers.

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Since nonlinearity in (1a) leads to nonlinear dependence of $v_\nu(t + \Delta t)$ on the components of $v_{\nu-r}(t),\ldots,v_{\nu+p}(t)$, we are unable to be more specific, at this stage, about the structure of the scheme S. However, we assume that S is L^2-stable, in case it is applied to the pure initial value problem for $-\infty < x < \infty$.

Usually, $r > 0$, so it is impossible to approximate (1) by (2) without specifying boundary values at r grid points in some left neighborhood of the boundary $x = 0$. Thus, we admit boundary conditions of the form

$$(3) \qquad v_\mu(t) = \sum_{j=1}^{s} c_j v_{\mu+j}(t), \qquad \mu = 0,\ldots,-r+1,$$

where the coefficients c_j and $s \geq 1$ are fixed. That is, having the values $v_\nu(t)$, $\nu \geq 1$, computed by the basic scheme (2), we proceed, at each time step, by using (3) to determine $v_\mu(t)$, $\mu = 0,-1,\ldots,-r+1$, in that order.

A natural way to choose the boundary conditions in (3) would be to employ extrapolation of degree $s - 1$ -- a procedure which is of accuracy of order s. More explicitly, we extrapolate from $v_1(t),\ldots,v_s(t)$ to $v_0(t)$; then from $v_0(t),\ldots,v_{s-1}(t)$ to $v_{-1}(t)$, etc. With the use of Stirling's extrapolation formula, (3) becomes

$$(4) \qquad v_\mu(t) = \sum_{j=1}^{s} \binom{s}{j}(-1)^{j+1} v_{\mu+j}(t), \qquad \mu = 0,\ldots,-r+1.$$

The main purpose of this note is to study the influence of boundary extrapolation on the stability of the numerical algorithm. This question is discussed in Section 2, where we consider a scalar linear conservation law which we approximate by a dissipative scheme. In this simple case it is shown that boundary extrapolation maintains stability. This result, which was first suggested by Kreiss, [5], was proven explicitely in [1], using Kreiss' theory, [6], for dissipative approximations of mixed initial-boundary value problems.

In fact, the above assertion is an immediate corollary of a forthcoming work by Goldberg and Tadmor, [2], which provides stability criteria for some general families of boundary conditions, including those presented in (3).

Finally, in Section 3, the Lax-Wendroff scheme, [7], and a new 5-point dissipative approximation by Gottlieb and Turkel, [3], are applied to a test problem. The numerical results support Gustafsson's rate-of-convergence theory, [4], by showing that the accuracy order of the basic scheme is maintained, if the extrapolation at the boundary is of the same order. The computation were carried out at the Campus Computing Network of the University of California at Los Angeles.

2. STABILITY ANALYSIS. From now on we restrict attention to the linear, scalar version of (1), namely to the initial value problem

$$\partial u/\partial t + a\partial u/\partial x = 0; \quad a = \text{const.}; \quad x \geq 0,\ t \geq 0; \quad u(x,0) = f(x), \tag{5}$$

which is well-posed if and only if $a < 0$.

Our explicit approximation in (2) becomes

$$\begin{gathered} v_\nu(t + \Delta t) = Qv_\nu(t), \qquad \nu = 1,2,3,\ldots, \\ Q = \sum_{j=-r}^{p} a_j E^j, \quad Ev_\nu = v_{\nu+1}, \quad r > 0, \end{gathered} \tag{6}$$

where the constants a_j depend on a and on the fixed ratio $\lambda = \Delta t/\Delta x$, and initial values are determined by

$$v_\nu(0) = f_\nu, \qquad \nu = 1,2,3,\ldots \ .$$

The assumption of dissipativity is that for some $\delta > 0$ and natural ω, the <u>amplification</u> <u>factor</u> of the scheme,

$$\hat{Q}(\xi) = \sum_{j=-r}^{p} a_j e^{ij\xi}, \qquad -\pi \leq \xi < \pi,$$

satisfies

$$|\hat{Q}(\xi)| \leq 1 - \delta|\xi|^{2\omega}, \qquad \forall\, |\xi| \leq \pi.$$

Thus, it is evident that $\hat{Q}(\xi)$ is now power bounded (by 1), which is well known to be equivalent to the (strong) stability of our basic scheme.

Introducing the boundary conditions (4), the concept of stability

becomes considerably more complicated, and we review it briefly. Let $H \equiv H(\Delta x)$ be the space of all grid functions, $w = \{w_\nu\}_{\nu=-r+1}^{\infty}$, which satisfy $\Sigma_{-r+1}^{\infty}|w_\nu|^2 < \infty$ and fulfill the boundary conditions in (4). If inner product and norm are defined by

$$(v,w) = \Delta x \sum_{\nu=-r+1}^{\infty} v_\nu \bar{w}_\nu, \qquad \|w\|^2 = (w,w),$$

then H becomes a discrete analogue of $L^2(0,x)$.

Having constructed H, we realize that our finite difference algorithm in (4) and (6) defines a linear, bounded operator, $G : H \to H$, such that the numerical solution v satisfies,

$$v(t + \Delta t) = Gv(t), \quad \text{for} \quad v(t) \in H.$$

Since

$$v(t) = G^m v(0) \quad \text{for} \quad t = m\Delta t, \quad m = 1,2,3,\ldots,$$

stability means that the powers of G are uniformly bounded, i.e., that for some constant K,

$$\|G^m\| \leq K, \qquad m = 1,2,3,\ldots \quad .$$

We are now ready to state the main result:

THEOREM 1. _Let the initial value problem_ (5) _be approximated by an arbitrary, dissipative_ (_stable_) _scheme of type_ (6), _which is complemented by boundary extrapolation of arbitrary order. Then, the overall numerical algorithm is stable._

The proof which is laid out in [1], is a direct but somewhat lengthy application of Kreiss' stability theory, [6], for dissipative schemes. As required by Kreiss' criterion, the problem was to show that the corresponding operator G has no eigenvalues z in the unit disk.

In a forthcoming paper, [2], Goldberg and Tadmor use Kreiss' theory to provide a particularly simple stability condition in the case where scheme (6) is augmented by boundary conditions of type (2). This condition is rephrased as follows:

THEOREM 2 (Goldberg, Tadmor). *Let* (6) *be an arbitrary, dissipative (stable) approximation, augmented by boundary conditions of type* (2), *then the overall algorithm is stable if*

$$\sum_{j=-r}^{p} c_j \kappa^j \neq 1, \quad \forall \kappa \quad \text{with} \quad |\kappa| < 1.$$

Theorem 2, which is actually independent of the basic scheme, yields Theorem 1 immediately. For, considering the boundary conditions in (4), we want to show that

$$\sum_{j=1}^{s} \binom{s}{j} (-1)^{j+1} \kappa^j \neq 1 \quad \text{for} \quad \kappa \quad \text{with} \quad |\kappa| < 1,$$

i.e., that for all κ with $|\kappa| < 1$,

$$(1 - \kappa)^s \equiv \sum_{j=0}^{s} \binom{s}{j} (-\kappa)^j \neq 0.$$

The last inequality holds, and Theorem 1 follows.

3. NUMERICAL RESULTS. Consider the test problem

(7) $$\partial u/\partial t - \partial u/\partial x = 0; \quad x \geq 0, \quad t \geq 0; \quad u(x,0) = \sin 2\pi x$$

whose analytic solution is

$$u(x,t) = \sin 2\pi(x + t).$$

The second order accurate Lax-Wendroff scheme (L-W), [7], is in this case

(8) $$v(t + \Delta t) = \frac{\lambda}{2}(\lambda - 1)v_{\nu-1}(t) + (1 - \lambda^2)v_\nu(t) + \frac{\lambda}{2}(\lambda + 1)v_{\nu+1}(t), \quad \lambda = \frac{\Delta t}{\Delta x},$$

and it is well known (e.g., [8, Chapter 12]) that dissipativity, and hence stability, are guaranteed if $\lambda < 1$.

In order to apply (8) to (7) we need to specify only one boundary value, $v_0(t)$, which according to (4), is given by

$$v_0(t) = \sum_{j=1}^{s} \binom{s}{j} (-1)^{j+1} v_j(t).$$

Here the accuracy of the boundary extrapolation is of orders s where s is arbitrary.

In Table 1 we compare the L-W results with the analytic solution at $t = 1$. The H-norm of the error was computed over the interval $0 \leq x \leq 1$, and is defined by

$$\text{(9)} \qquad \|e\|^2_{(0,1)} = \Delta x \sum_{\nu=0}^{J} [v_\nu(1) - u(\nu\Delta x, 1)]^2, \qquad J = 1/\Delta x.$$

Δx	s	m	$\|e\|_{(0,1)}$
.05	2	40	5.57 - 2
.025	2	80	1.39 - 2
.05	1	40	7.03 - 2
.025	1	80	2.23 - 2

Table 1. L-W results at $t = 1$;
$\lambda = 1/2$; $m = t/\Delta t$ is number of time steps.

Gustafsson, in his rate-of-convergence theory, [4], has discussed situations similar to the one under consideration. He has shown that in order to maintain the accuracy of the basic scheme, it is sufficient to employ boundary conditions of the same order of accuracy. Indeed, Table 1 suggests that L-W's second order accuracy is maintained if the boundary extrapolation is linear $(s = 2)$, but is reduced if $s = 1$.

A second example is concerned with a family of centered, 5-point, dissipative schemes by Gottlieb and Turkel (G-T), [3]. The family, given in (2.4) of [3], depends on two parameters α and σ. Choosing $\alpha = 1/2$, $\sigma = 1$, and linearizing, we obtain an approximation to (7) of the form

$$v_\nu(t + \Delta t) = -\frac{\lambda}{4}\left(\frac{\lambda}{2} - \frac{1}{3}\right)v_{\nu-2}(t) + \lambda\left(\lambda - \frac{2}{3}\right)v_{\nu-1}(t) + \left(1 - \frac{7}{4}\lambda^2\right)v_\nu(t)$$
$$+ \lambda\left(\lambda + \frac{2}{3}\right)v_{\nu+1}(t) - \frac{\lambda}{4}\left(\frac{\lambda}{2} + \frac{1}{3}\right)v_{\nu+2}(t), \qquad \lambda = \frac{\Delta t}{\Delta x},$$

where the dissipativity condition is $\lambda < \sqrt{2}/2$.

Now we need two boundary values which are given by

$$v_\mu(t) = \sum_{j=1}^{s} \binom{s}{j} (-1)^{j+1} v_{\mu+j}(t), \qquad \mu = 0, -1.$$

The error-norms in Table 2 are computed, as in the previous case.

over $0 \leq x \leq 1$, and in analogy to (8) are defined by

$$\|e\|^2_{(0,1)} = \Delta x \sum_{\nu=-1}^{J} [v_\nu(1) - u(\nu\Delta x,1)]^2, \qquad J = 1/\Delta x.$$

Δx	λ	s	m	$\|e\|_{(0,1)}$
.05	.5	4	40	1.85 - 2
.025	.25	4	160	1.16 - 3
.05	.5	3	40	2.42 - 2
.025	.25	3	160	1.92 - 3

Table 2. G-T results for $t = 1$.

Unlike the L-W scheme which is of second order accuracy both in time and space, the G-T approximation is of second order in time and fourth order in space. Since the boundary extrapolation is taken only with respect to the space variable, it should be expected that in order to maintain the fourth-order accuracy in x, we have to utilize cubic extrapolation ($s = 4$), regardless of the fact that G-T's accuracy in time is only of second order. This is reflected by the results of Table 2.

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